

The k -proper index of graphs*

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Abstract

A tree T in an edge-colored graph is a *proper tree* if any two adjacent edges of T are colored with different colors. Let G be a graph of order n and k be a fixed integer with $2 \leq k \leq n$. For a vertex set $S \subseteq V(G)$, a tree containing the vertices of S in G is called an *S -tree*. An edge-coloring of G is called a *k -proper coloring* if for every set S of k vertices in G , there exists a proper S -tree in G . The *k -proper index* of a nontrivial connected graph G , denoted by $px_k(G)$, is the smallest number of colors needed in a k -proper coloring of G . In this paper, some simple observations about $px_k(G)$ for a nontrivial connected graph G are stated. Meanwhile, the k -proper indices of some special graphs are determined, and for every pair of positive integers a, b with $2 \leq a \leq b$, a connected graph G with $px_k(G) = a$ and $rx_k(G) = b$ is constructed for each integer k with $3 \leq k \leq n$. Also, the graphs with k -proper index $n - 1$ and $n - 2$ are respectively characterized.

Keywords: coloring of graphs, k -proper index, characterization of graphs

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1 Introduction

In this paper, all graphs under our consideration are finite, undirected, connected and simple. For more notation and terminology that will be used in the sequel, we refer to [2], unless otherwise stated.

In 2008, Chartrand et al. [8] first introduced the concept of rainbow connection. Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \rightarrow \{1, 2, \dots, k\}$ ($k \in \mathbb{N}$) is defined, where adjacent edges may be colored with the same color. For any two vertices u and v of G , a path in G connecting u and v is abbreviated as a uv -path. A uv -path P is a *rainbow uv -path* if no two edges of P are colored with the same color. The graph G is *rainbow connected* (with respect to c) if G contains a rainbow uv -path for every two vertices u and v , and the coloring c is called a *rainbow coloring* of G . If k colors are used, then c is a *rainbow k -coloring*. The minimum k for which there exists

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a rainbow k -coloring of the edges of G is the *rainbow connection number* of G , denoted by $rc(G)$. The topic of rainbow connection is fairly interesting and numerous relevant papers have been written. For more details see a survey [23] and a book [24].

Subsequently, a series of generalizations of rainbow connection number were proposed. The k -rainbow index is one of them. An edge-colored tree T is a *rainbow tree* if no two edges of T are assigned the same color. Let G be a nontrivial connected graph of order n and let k be an integer with $2 \leq k \leq n$. A k -rainbow coloring of G is an edge coloring of G having the property that for every set S of k vertices of G , there exists a rainbow tree T in G such that $S \subseteq V(T)$. The minimum number of colors needed in a k -rainbow coloring of G is the k -rainbow index of G . These concepts were introduced by Chartrand et al. in [9], and were further studied in [4, 5, 10, 21, 22, 26].

In addition, a natural extension of the rainbow connection number is the proper connection number, which was introduced by Borożan et al. in [3]. A path in an edge-colored graph is said to be *properly edge-colored (or proper)*, if every two adjacent edges differ in color. An edge-colored graph G is k -proper connected if any two vertices are connected by k internally pairwise vertex-disjoint proper paths. The k -proper connection number of a k -connected graph G , denoted by $pc_k(G)$, is defined as the smallest number of colors that are needed in order to make G k -proper connected. In particular, when $k = 1$, the 1-proper connection number is abbreviated as proper connection number and written as $pc(G)$. For more results, we refer to [1, 12–15, 18, 25].

Inspired by the k -rainbow index and the proper connection number, a natural idea is to introduce the concept of k -proper index. A tree T in an edge-colored graph is a *proper tree* if any two adjacent edges of T are colored with different colors. Let G be a graph of order n and k be a fixed integer with $2 \leq k \leq n$. For a vertex set $S \subseteq V(G)$, a tree containing the vertices of S in G is called an S -tree. An edge-coloring of G is called a k -proper coloring if for every set S of k vertices in G , there exists a proper S -tree in G . The k -proper index of a nontrivial connected graph G , denoted by $px_k(G)$, is the smallest number of colors needed in a k -proper coloring of G . By definition, $px_2(G)$ is precisely the proper connection number $pc(G)$ for any nontrivial graph G . As a variety of nice results about $pc(G) = px_2(G)$ have been obtained, we in this paper only study $px_k(G)$ for $3 \leq k \leq n$.

The paper is organized as follows: In Section 2, some simple observations about $px_k(G)$ for a nontrivial graph G are stated. Meanwhile, certain necessary lemmas are also listed. In Section 3, the k -proper indices of some special graphs are determined. And for every pair of positive integers a, b with $2 \leq a \leq b$, a connected graph G with $px_k(G) = a$ and $rx_k(G) = b$ is constructed for each integer k with $3 \leq k \leq n$. In Section 4, the graphs with k -proper index $n - 1$ and $n - 2$ are characterized, respectively.

2 Preliminaries

We in this section state some observations about $px_k(G)$ for a nontrivial graph G . Also, certain necessary lemmas are listed.

For a graph G with order $n \geq 3$, it follows from the definition that

$$(*) \quad pc(G) = px_2(G) \leq px_3(G) \leq px_4(G) \leq \cdots \leq px_n(G).$$

This simple property will be used frequently later.

Since any k -proper coloring of a spanning subgraph must be a k -proper coloring of its supergraph. Then there exists a fundamental proposition about spanning subgraphs.

Proposition 1. *If G is a nontrivial connected graph of order $n \geq 3$ and H is a connected spanning subgraph of G , then $px_k(G) \leq px_k(H)$ for any k with $3 \leq k \leq n$. In particular, $px_k(G) \leq px_k(T)$ for every spanning tree T of G .*

It has been seen in [9] that $rx_k(G) \leq n - 1$ for any graph G with order $n \geq 3$ and any integer k with $3 \leq k \leq n$. Since a rainbow tree must be a proper tree, then obviously $px_k(G) \leq rx_k(G) \leq n - 1$. Moreover, this simple upper bound is sharp, the graphs with $px_k(G) = n - 1$ will be characterized in Section 4.

For any nontrivial graph G , $\chi'(G)$ denotes the edge-chromatic number of G . It is well-known that either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$ by Vizing's Theorem, where $\Delta(G)$, or simply Δ , is the maximum degree of G . Accordingly, a natural upper bound of $px_k(G)$ with respect to these parameters follows.

Proposition 2. *Let G be a graph with order $n \geq 3$, maximum degree $\Delta(G)$ and edge-chromatic number $\chi'(G)$. Then for each integer k with $3 \leq k \leq n$, we have*

$$px_k(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

In addition, there is a classical result about the edge-chromatic number of a graph, which will be useful in the next section.

Lemma 1 ([2]). *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

For arbitrary k ($k \geq 3$) vertices of a nontrivial graph G , any tree T containing these vertices must contain internal vertices. While for any proper tree T , there must be $d(u)$ distinct colors assigned to the incident edges of each vertex u in T , where $d(u)$ denotes the degree of u in T . Hence, the incident edges of any internal vertex must be assigned with at least two distinct colors to make T proper. Then the following trivial lower bound is immediate.

Proposition 3. *For arbitrary graph G with order $n \geq 3$, we have*

$$px_k(G) \geq 2$$

for any integer k with $3 \leq k \leq n$.

Remark: The above lower bound of $px_k(G)$ is sharp since there exist many graphs satisfying $px_k(G) = 2$, as shown in Section 3. Further, we believe that it will be interesting to characterize all graphs with k -proper index 2 for specific values of k .

In any graph G , a path (resp. cycle) that contains every vertex of G is called a *Hamilton path* (resp. *Hamilton cycle*) of G . A graph is *traceable* if it contains a Hamilton path, and a graph is *hamiltonian* if it contains a Hamilton cycle. The following is an immediate consequence of these definitions as well as Proposition 3.

Proposition 4. *If G is a traceable graph with order $n \geq 3$, then $px_k(G) = 2$ for each integer k with $3 \leq k \leq n$.*

As mentioned before, characterizing all graphs with k -proper index 2 for specific values of k would be interesting. While for the cases of $k = n$ and $k = n - 1$, there are two basic results that can be mentioned.

Observation 1. *If a graph G of order n satisfies $px_n(G) = 2$, namely, $px_k(G) = 2$ for each k with $3 \leq k \leq n$. Then G is a traceable graph.*

Observation 2. *If a graph G of order n satisfies $px_k(G) = 2$ for each k with $3 \leq k \leq n - 1$. Then $px_n(G) = 2$ if and only if G is traceable. Otherwise, $px_n(G) = 3$.*

It is well known that if G is a simple graph with order $n \geq 3$ and minimum degree $\delta \geq \frac{n}{2}$, then G is hamiltonian. Whereupon a direct corollary follows.

Corollary 1. *If G is a simple graph with order $n \geq 3$ and minimum degree $\delta \geq \frac{n}{2}$, then $px_k(G) = 2$ for each integer k with $3 \leq k \leq n$.*

In [9], Chartrand et al. derived the k -rainbow index of a nontrivial tree, which will be helpful in the next section.

Lemma 2 ([9]). *Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$,*

$$rx_k(T) = n - 1.$$

In [3], Borożan et al. established the proper connection number of trees.

Lemma 3 ([3]). *If G is a tree then $pc(G) = \Delta(G)$.*

At the end of this section, we recall several notations required in the subsequent sections.

Let $E' \subseteq E(G)$ be a set of edges of a graph G , then $G[E']$ is the subgraph of G induced by E' . If e is an edge of G , then $G - e$ denotes the graph obtained from G by only deleting the edge e . If G is not complete, denote by $G + e$ the graph obtained from G by the addition of e , where e is an edge connecting two nonadjacent vertices of G .

3 The k -proper indices of special graphs

In this section, we determine the k -proper indices of complete graphs, cycles, wheels, trees and unicyclic graphs. Moreover, the independence of $px_k(G)$ and $rx_k(G)$ is given by a brief theorem.

It has been seen that if G is a traceable graph, then $px_k(G) = 2$. Obviously, the complete graphs, cycles and wheels are all traceable, thus the k -proper indices of these graphs are direct consequences of Proposition 4.

Theorem 1. *Let K_n , C_n and W_n be a complete graph, a cycle and a wheel with n ($n \geq 3$) vertices, respectively. Then for any integer k with $3 \leq k \leq n$, we have*

$$px_k(K_n) = px_k(C_n) = px_k(W_n) = 2.$$

Now we determine the k -proper index for a nontrivial tree.

Theorem 2. *If T is a tree of order $n \geq 3$, then for each integer k with $3 \leq k \leq n$,*

$$px_k(T) = \Delta(T).$$

Proof. Firstly, since T is bipartite, then $px_k(T) \leq \chi'(T) = \Delta(T)$ for $3 \leq k \leq n$ by Proposition 2 and Lemma 1. On the other hand, according to Ineq. (*) and Lemma 3, $px_k(T) \geq pc(T) = \Delta(T)$ holds naturally for $3 \leq k \leq n$. Therefore, we arrive at $px_k(T) = \Delta(T)$ for any integer k with $3 \leq k \leq n$. ■

Combine with Proposition 1 and Theorem 2, one can check that the following proposition holds.

Proposition 5. *For any graph G with order $n \geq 3$ and any integer k with $3 \leq k \leq n$, we have*

$$px_k(G) \leq \min\{\Delta(T) : T \text{ is a spanning tree of } G\}.$$

Since $\Delta(T) \leq \Delta(G)$ for any spanning tree T of G . Then the upper bound in Proposition 2 can be replaced by $\Delta(G)$.

Proposition 6. *Let G be a graph with order $n \geq 3$ and maximum degree $\Delta(G)$, then*

$$px_k(G) \leq \Delta(G)$$

for each integer k with $3 \leq k \leq n$.

Remark: The above upper bound of $px_k(G)$ is sharp since the equality holds apparently for arbitrary nontrivial tree.

In order to get the k -proper index of a unicyclic graph, an assistant lemma is presented.

Lemma 4. *Let G be a graph of order $n \geq 3$ containing bridges and v be any vertex of G . Denote by $b(v)$ the number of bridges incident with v . Set $b = \max\{b(v) : v \in V(G)\}$. Then for each integer k with $3 \leq k \leq n$, we have $px_k(G) \geq b$.*

Proof. Since for $3 \leq k \leq n$, it has been seen from Ineq. (*) that $px_k(G) \geq px_3(G)$. Then we should only prove the case when $k = 3$. Since $px_3(G) \geq 2$ by Proposition 3, the result is trivial when $b = 1$ or 2 . Thus we may assume that $b \geq 3$. Suppose that $b(u) = b = \max\{b(v) : v \in V(G)\}$ for some vertex u . Let $F = \{uw_1, uw_2, \dots, uw_b\}$ be the set of bridges incident with u . Set $A = \{u, w_1, w_2, \dots, w_b\}$. For any 3-set $S = \{w_i, w_j, u\} \subseteq A$, where $i, j \in \{1, 2, \dots, b\}$ and $i \neq j$, every S -tree T must contain the edges uw_i and uw_j . Hence, the edges uw_i and uw_j receive distinct colors to make T proper. Which implies that the edges uw_1, uw_2, \dots, uw_b need b distinct colors in any 3-proper coloring of G . Therefore, $px_3(G) \geq b$. This completes the proof. ■

With the aid of Lemma 4, now we are able to deal with the k -proper index for a unicyclic graph.

Theorem 3. *Let G be a unicyclic graph of order $n \geq 3$, and maximum degree $\Delta(G)$. Then, for each integer k with $3 \leq k \leq n$,*

$$px_k(G) = \Delta(G) - 1$$

when G contains at most two vertices having maximum degree such that the vertices with maximum degree are all in the unique cycle of G and these two vertices (if both exist) are adjacent;

Otherwise,

$$px_k(G) = \Delta(G).$$

Proof. Note that when $G = C_n$, it follows from Theorem 1 that $px_k(G) = px_k(C_n) = 2 = \Delta(G)$ for $3 \leq k \leq n$. Thus in the following we assume that G is not a cycle. And assume the vertices in the unique cycle of G are u_1, u_2, \dots, u_g . Besides, keep in mind that $px_k(G) \leq \Delta(G)$ for $3 \leq k \leq n$, which will be used later. As before, denote by $b(v)$ the number of bridges incident with the vertex v . The discussion is divided into three cases.

Case 1. At first, assume that G contains a vertex, say u , satisfying

- (1) the degree of u is $d(u) = \Delta(G)$.
- (2) u is not in the cycle of G .

Then evidently the incident edges of u are all bridges, i.e., $b(u) = d(u) = \Delta(G)$. According to Lemma 4, we directly have $px_k(G) \geq b(u) = \Delta(G)$ for $3 \leq k \leq n$. Meanwhile, Proposition 6 guarantees $px_k(G) \leq \Delta(G)$ for $3 \leq k \leq n$. Accordingly, we get $px_k(G) = \Delta(G)$ for each integer k with $3 \leq k \leq n$ in this case.

By Case 1, if such a vertex u exists in G , then we always have $px_k(G) = \Delta(G)$ for each integer k with $3 \leq k \leq n$. To avoid redundant presentation, we in the following suppose that G contains no such vertices.

Case 2. Now assume G simultaneously satisfies

- (3) G contains at most two vertices having maximum degree;
- (4) the vertices with maximum degree are all in the unique cycle of G ;
- (5) these two vertices (if both exist) are adjacent in G .

Then without loss of generality, suppose that $d(u_1) = \Delta(G)$, $d(u_2) \leq \Delta(G)$ and $d(u) < \Delta(G)$ for any other vertex u . Moreover, suppose that the neighbors of u_1 are $v_1, v_2, \dots, v_{\Delta(G)-2}, v_{\Delta(G)-1} = u_2$ and $v_{\Delta(G)} = u_g$. Thereupon, in any 3-proper coloring c of G , based on the proof of Lemma 4, the edges u_1v_i with $i \in \{1, 2, \dots, \Delta(G) - 2\}$ are assigned with $\Delta(G) - 2$ distinct colors since they are all bridges incident with u_1 . Without loss of generality, suppose that $c(u_1v_1) = 1, c(u_1v_2) = 2, \dots, c(u_1v_{\Delta(G)-2}) = \Delta(G) - 2$. Further, we claim that at least a new color is used by the edges u_1u_2 and u_1u_g . For otherwise, suppose that $c(u_1u_2) = i$ and $c(u_1u_g) = j$ with $i, j \in \{1, 2, \dots, \Delta(G) - 2\}$. If $i = j$, then there exists no proper tree containing the vertices u_1, u_2 and v_i , a contradiction. If $i \neq j$, then there exists no proper tree containing the vertices v_i, v_j and u_2 , again a contradiction. Therefore, at least $\Delta(G) - 2 + 1 = \Delta(G) - 1$ different colors are used by c . It follows that $px_3(G) \geq \Delta(G) - 1$. Thus, Ineq.(*) deduces that $px_k(G) \geq px_3(G) \geq \Delta(G) - 1$ for each integer k with $3 \leq k \leq n$. On the other hand, obviously $G - u_1u_2$ is a spanning tree of G with maximum degree $\Delta(G) - 1$. By Theorem 2, we know that $px_k(G - u_1u_2) = \Delta(G - u_1u_2) = \Delta(G) - 1$ for $3 \leq k \leq n$. Consequently, $px_k(G) \leq px_k(G - u_1u_2) = \Delta(G) - 1$ based on Proposition 1. To sum up, we obtain $px_k(G) = \Delta(G) - 1$ for each integer k with $3 \leq k \leq n$ in this case.

Case 3. Finally, we discuss the case when G contains at least two vertices u_i and u_j such that

- (6) $d(u_i) = d(u_j) = \Delta(G)$;
- (7) both u_i and u_j are in the cycle of G ;
- (8) u_i and u_j are not adjacent in G .

Then we claim that $px_3(G) \geq \Delta(G)$. Assume to the contrary, $px_3(G) \leq \Delta(G) - 1$. Let c' be a 3-proper coloring of G using colors from $\{1, 2, \dots, \Delta(G) - 1\}$. Let the neighbors of u_i be $w_1, w_2, \dots, w_{\Delta(G)-2}, w_{\Delta(G)-1} = u_{i-1}, w_{\Delta(G)} = u_{i+1}$, and the neighbors of u_j be $z_1, z_2, \dots, z_{\Delta(G)-2}, z_{\Delta(G)-1} = u_{j-1}, z_{\Delta(G)} = u_{j+1}$. Similarly to Case 2, the edges u_iw_t with $t \in \{1, 2, \dots, \Delta(G) - 2\}$ are assigned with $\Delta(G) - 2$ distinct colors. Without loss of generality, suppose that $c'(u_iw_1) = 1, c'(u_iw_2) = 2, \dots, c'(u_iw_{\Delta(G)-2}) = \Delta(G) - 2$. Thus, either $c'(u_iu_{i-1}) = c'(u_iu_{i+1}) = \Delta(G) - 1$, or there exists at least one edge between u_iu_{i-1} and u_iu_{i+1} , say u_iu_{i-1} , such that $c'(u_iu_{i-1}) = x_1$ with $x_1 \in \{1, 2, \dots, \Delta(G) - 2\}$. Similarly, the edges u_jz_t with $t \in \{1, 2, \dots, \Delta(G) - 2\}$ also receive $\Delta(G) - 2$ distinct colors. And for the edges u_ju_{j-1} and u_ju_{j+1} , either $c'(u_ju_{j-1}) = c'(u_ju_{j+1})$, or there exists at least one of them, say u_ju_{j+1} , such that $c'(u_ju_{j+1}) = c'(u_jz_{x_2})$ with $x_2 \in \{1, 2, \dots, \Delta(G) - 2\}$.

- (i) If $c'(u_iu_{i-1}) = c'(u_iu_{i+1})$ and $c'(u_ju_{j-1}) = c'(u_ju_{j+1})$, then there exists no proper tree containing the vertices u_{i-1}, u_{i+1} and w_1 , a contradiction.
- (ii) If $c'(u_iu_{i-1}) = c'(u_iu_{i+1})$ and $c'(u_ju_{j+1}) = c'(u_jz_{x_2})$ with $x_2 \in \{1, 2, \dots, \Delta(G) - 2\}$, then there exists no proper tree containing the vertices u_{j+1}, u_j and z_{x_2} , a contradiction.

- (iii) If $c'(u_i u_{i-1}) = x_1$ with $x_1 \in \{1, 2, \dots, \Delta(G) - 2\}$ and $c'(u_j u_{j-1}) = c'(u_j u_{j+1})$, then there exists no proper tree containing the vertices u_{i-1} , u_i and w_{x_1} , a contradiction.
- (iv) If $c'(u_i u_{i-1}) = x_1$ with $x_1 \in \{1, 2, \dots, \Delta(G) - 2\}$ and $c'(u_j u_{j+1}) = c'(u_j z_{x_2})$ with $x_2 \in \{1, 2, \dots, \Delta(G) - 2\}$, then there exists no proper tree containing the vertices w_{x_1} , u_{i-1} and z_{x_2} , a contradiction.

In summary, we verify that $px_3(G) \geq \Delta(G)$, which deduces that $px_k(G) \geq px_3(G) \geq \Delta(G)$ for $3 \leq k \leq n$. Combine with $px_k(G) \leq \Delta(G)$ for $3 \leq k \leq n$, we at last arrive at $px_k(G) = \Delta(G)$ for each integer k with $3 \leq k \leq n$ in this case.

The proof of this theorem is finished. ■

We conclude this section with a simple theorem to address the independence of $px_k(G)$ and $rx_k(G)$.

Theorem 4. *For every pair of positive integers a, b with $2 \leq a \leq b$, there exists a connected graph G such that $px_k(G) = a$ and $rx_k(G) = b$ for each integer k with $3 \leq k \leq n$.*

Proof. For each pair of positive integers a, b with $2 \leq a \leq b$, let G be a nontrivial tree with order $n = b + 1$ and maximum degree $\Delta(G) = a$. The existence of such a tree is guaranteed by $2 \leq a \leq b$. Then based on Theorem 2 and Lemma 2, we know that $px_k(G) = \Delta(G) = a$ and $rx_k(G) = n - 1 = b$ for each integer k with $3 \leq k \leq n$. The proof is thus complete. ■

4 Graphs with k -proper index $n - 1, n - 2$

In this section, we are going to characterize the graphs whose k -proper index equals to $n - 1$ and $n - 2$, respectively, where $3 \leq k \leq n$. First of all, we give the following lemma that will be used in the sequel.

Lemma 5. *For $n \geq 5$, let S_n^+ be the graph obtained by adding a new edge to the n -vertices star S_n , and S_n^{++} be the graph obtained by adding a new edge to S_n^+ . Then we have $px_k(S_n^{++}) \leq n - 3$ for each integer k with $3 \leq k \leq n$.*

Proof. Let $V(S_n^+) = V(S_n^{++}) = \{u, v_1, v_2, \dots, v_{n-1}\}$. Without loss of generality, set $d_{S_n^+}(u) = d_{S_n^{++}}(u) = n - 1$ and $d_{S_n^+}(v_1) = d_{S_n^+}(v_2) = 2$. Further, let e be the edge of S_n^{++} added to S_n^+ . We split the remaining proof into the following two cases depending on the position of e .

Case 1. The edges e and $v_1 v_2$ are vertex-disjoint. Without loss of generality, let $e = v_3 v_4$. Then, $G' = G - uv_1 - uv_3$ is a spanning tree of S_n^{++} with maximum degree $n - 1 - 2 = n - 3$. It follows from Theorem 2 that $px_k(G') = \Delta(G') = n - 3$ for $3 \leq k \leq n$. Hence, Proposition 1 deduces that $px_k(S_n^{++}) \leq px_k(G') = n - 3$ for each integer k with $3 \leq k \leq n$.

Case 2. The edges e and v_1v_2 have a common vertex. Without loss of generality, let $e = v_2v_3$. Then, $G'' = G - uv_2 - uv_3$ is a spanning tree of S_n^{++} with maximum degree $n - 1 - 2 = n - 3$. Similarly, $px_k(G'') = \Delta(G'') = n - 3$ for $3 \leq k \leq n$. Hence, we can also get $px_k(S_n^{++}) \leq px_k(G'') = n - 3$ for each integer k with $3 \leq k \leq n$.

Combining the above two cases, now the lemma follows. \blacksquare

Theorem 5. *Let G be a connected graph of order n ($n \geq 4$). Then for each integer k with $3 \leq k \leq n$, $px_k(G) = n - 1$ if and only if $G \cong S_n$, where S_n is the star of order n .*

Proof. Firstly, if $G \cong S_n$, then by Theorem 2, we directly obtain $px_k(G) = px_k(S_n) = \Delta(S_n) = n - 1$ for $3 \leq k \leq n$.

Conversely, suppose G is a graph with $px_k(G) = n - 1$ for each integer k with $3 \leq k \leq n$. Since $n - 1 = px_k(G) \leq \Delta(G)$ by Proposition 6, meanwhile $\Delta(G) \leq n - 1$ holds for any simple graph with order n . Then, $\Delta(G) = n - 1$. The hypothesis is true if $G \cong S_n$. If $G \not\cong S_n$, let u be a vertex of G with $d(u) = \Delta(G) = n - 1$. Let $V(G) \setminus u = \{v_1, v_2, \dots, v_{n-1}\}$ denote the set of the remaining vertices in G . Since $G \not\cong S_n$, there exist at least two vertices, say v_1 and v_2 , such that they are adjacent in G . Set $G' = G[\bigcup_{i=1}^{n-1} uv_i] + v_1v_2$. Then, as $n \geq 4$, G' is a unicyclic graph satisfying the conditions in Case 2 of Theorem 3 with maximum degree $n - 1$. Hence, $px_k(G') = \Delta(G') - 1 = n - 2$ based on the result of Theorem 3. Apparently, G' is a spanning subgraph of G , therefore $px_k(G) \leq px_k(G') = n - 2$ for $3 \leq k \leq n$ according to Proposition 1, contradicting our assumption that $px_k(G) = n - 1$. Consequently, $G \cong S_n$.

The proof is thus complete. \blacksquare

Remark: If G is a graph with order $n = 3$, then one can check that $px_3(G) = n - 1 = 2$ if and only if $G \cong S_3$ or $G \cong C_3$.

Theorem 6. *Let G be a connected graph of order n ($n \geq 5$). Then for each integer k with $3 \leq k \leq n$, $px_k(G) = n - 2$ if and only if $G \cong S_n^+$ or G_0 , where S_n^+ is defined in Lemma 5 and G_0 is shown in Figure 1.*

Proof. On the one hand, if $G \cong S_n^+$, then G is a unicyclic graph with maximum degree $n - 1$ satisfying the conditions in Case 2 of Theorem 3. Thus $px_k(G) = px_k(S_n^+) = \Delta(G) - 1 = n - 2$ for $3 \leq k \leq n$. If $G \cong G_0$, then G is a tree with order $n \geq 5$ and maximum degree $n - 2$. Accordingly, by Theorem 2, $px_k(G) = px_k(G_0) = \Delta(G) = n - 2$ for $3 \leq k \leq n$.

On the other hand, if $px_k(G) = n - 2$, then by Proposition 6, $\Delta(G) \geq px_k(G) = n - 2$, which means that $\Delta(G) = n - 2$ or $n - 1$. The remaining proof is divided into two cases depending on the value of $\Delta(G)$.

Case 1. $\Delta(G) = n - 1$.

In this case, since $px_k(S_n) = n - 1$ for $3 \leq k \leq n$, as shown before, then G must contain

S_n^+ as a connected spanning subgraph. If $G \cong S_n^+$, we have known that $px_k(S_n^+) = n-2$ for $3 \leq k \leq n$. Now suppose $G \not\cong S_n^+$. Then there exists a connected spanning subgraph with the form of S_n^{++} in G . Applying Proposition 1 together with Lemma 5, we arrive at $px_k(G) \leq px_k(S_n^{++}) \leq n-3$, a contradiction. Hence, $G \cong S_n^+$ in this case.

Case 2. $\Delta(G) = n-2$.

Then G_0 must be a connected spanning subgraph of G . If $G \cong G_0$, then $px_k(G_0) = n-2$ for $3 \leq k \leq n$. If $G \not\cong G_0$, then there exists at least one edge $e \in E(G) \setminus E(G_0)$. Thus, G contains a connected spanning subgraph isomorphic to G_1 , G_2 or G_3 , where G_1 , G_2 and G_3 are shown in Figure 1. Clearly, one can check that G_1 , G_2 and G_3 are all unicyclic graphs with maximum degree $n-2$ satisfying the conditions in Case 2 of Theorem 3. Thereupon, by Theorem 3 as well as Proposition 1, we directly get that $px_k(G) \leq px_k(G_i) = \Delta(G_i) - 1 = n-3$ for $3 \leq k \leq n$ and $i = 1, 2$ or 3 , which is a contradiction. Accordingly, $G \cong G_0$ in this case.

In summary, if $px_k(G) = n-2$ for $3 \leq k \leq n$, then $G \cong S_n^+$ or $G \cong G_0$. And the proof of this theorem is complete. \blacksquare

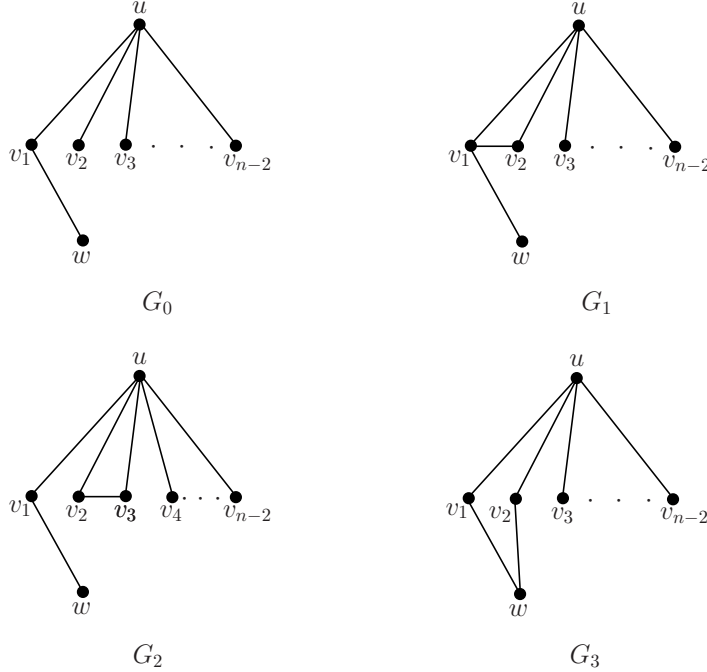


Figure 1: The graphs G_i for $i = 0, 1, 2, 3$.

Remark: When $n = 4$, except for the star S_4 , other connected graphs with order 4 are all traceable. Then by Proposition 4, the k -proper indices of these graphs equal to $2 = n-2$ for each integer k with $3 \leq k \leq 4$. While for the star S_4 , we know that $px_k(S_4) = 3$ for $3 \leq k \leq 4$. Consequently, we can easily claim that if G is a connected graph of order $n = 4$, then for each integer k with $3 \leq k \leq 4$, $px_k(G) = n-2 = 2$ if

and only if $G \not\cong S_4$.

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